Number Systems

What is a number? There's a lot more to this question than may first be obvious. The modern conception of number is way different than how it has been understood historically.

We learn numbers through counting objects. The simplest numbers are these "counting numbers", called the Natural Numbers, or $\mathbb{N}$

$$\mathbb{N} =: \{x \mid x = 1, 2, 3, 4, 5 \ldots\}$$

Indian mathematicians were the first to use a symbol to represent a lack of quantity. They used zero, 0, to represent an empty space with nothing inside. The great advance here is that the lack of number is itself a number. Arabic mathematicians brought these ideas to Europe after c. 1200.

The whole numbers are the natural numbers & 0.

$$\mathbb{W} =: \{x \mid x = 0 \text{ or } x \in \mathbb{N}\}$$

Chinese mathematicians used negative numbers as early as 300ad, and

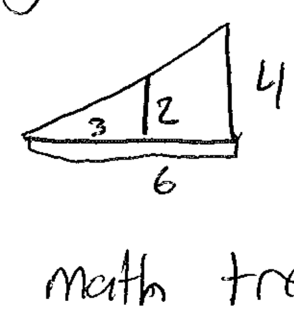
Arabic mathematicians worked out their properties from c900–1200. Europeans regarded negative numbers with suspicion, and called them "absurd" numbers until the development of Calculus with G.W. Leibniz.

The Integers are positive & negative whole numbers.

$$\mathbb{Z} = \{x \mid x \in \mathbb{W} \text{ or } -x \in \mathbb{W}\}$$

(The Z is from German, "Zahlen" = "numbers")

Greek mathematicians did not think of fractions as numbers, rather they thought of fractions as ratios of 2 numbers. So, "2/3" isn't a single number, it represents a comparison of the numbers 2 & 3. Remember that geometry = math for the Greeks. Fractions for them are like similar triangles for us.



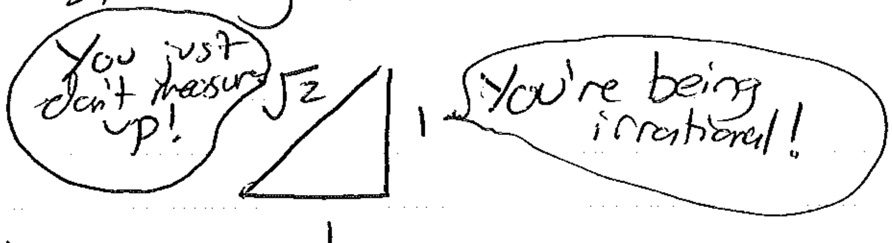
$$\frac{2}{3} = \frac{4}{6}.$$

Modern math treats fractions as numbers in and of themselves. They aren't just a comparison of 2 quantities; they are numbers.

Rational Numbers: All ratios of integers except where the denominator is zero.

$$\mathbb{Q} = \{ x \mid x = \frac{a}{b},\ a,b \in \mathbb{Z},\ b \neq 0 \}$$

The Greeks (and especially the Divine Brotherhood of the Pythagoreans) really believed that all lengths were rational. The story goes that Hipasos, one of the D.B.P., was drowned for demonstrating that $\sqrt{2}$ is irrational.



Real Numbers: $\mathbb{R} = \{ x \mid x \in \mathbb{Q} \text{ or } x \text{ is irrational} \}$

I'll prove to you that $\sqrt{2}$ is irrational and that $\log_2 3$ is irrational.

I'll use "proof by contradiction", where you assume something false and derive a contradiction.

1) Want to prove: $\sqrt{2}$ is irrational

Assume the opposite: $\sqrt{2} = \frac{a}{b}$, a/b is fully simplified

So $(\sqrt{2})^2 = (\frac{a}{b})^2 \rightarrow 2 = \frac{a^2}{b^2} \rightarrow 2b^2 = a^2$

Thus a is an even number. So $a = 2x$.

Substitute: $2b^2 = (2x)^2 \rightarrow 2b^2 = 4x^2$

So $b^2 = 2x^2$, and b is <u>also</u> even.

This contradicts that a/b is simplified. Thus $\sqrt{2}$ is irrational, QED.

2) Want to prove: $\log_2 3$ is irrational

Assume: $\log_2 3 = a/b$, a/b is simplified.

So $2^{a/b} = 3$, & $2^a = 3^b$.

But no power of 2 is ever divisible by 3. So $2^a \neq 3^b$. $\log_2 3$ can't equal a fraction.

Complex Numbers: All real numbers and their sums with imaginary numbers

$$\mathbb{C} = \{x \mid x = a + bi,\ a, b \in \mathbb{R}\}$$

Rafael Bombelli used imaginary numbers to solve polynomial equations in 1572. He was able to solve many equations that could not otherwise be solved, but other mathematicians derided his method as invalid. They called his numbers "imaginary" as a derogatory term.

It took a century for their use to be widely accepted. Although they are imaginary numbers, their application led to advances in many sciences.

Each number system is contained in a "bigger" one.

$$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$