Remainder Thm.

If $p(c)=K$, then $p(x)=q(x)(x-c)+K$.

Use this theorem & substitution to find the equation of polynomials passing through arbitrary points.

Please try to solve the problem independently before reading my solutions.

The problems will get nastier as we move through this sheet.

1) Find an $n=2$ polynomial passing through $(0,1)$, $(1,3)$, $(2,9)$

So $p(0)=1 \longrightarrow p(x)=q(x)(x-0)+1$

$p(1)=3 \longrightarrow 3=q(1)(1)+1$

$-1 \qquad -1$

$2=q(1)$

$q(1)=2 \longrightarrow q(x)=r(x)(x-1)+2$

p has $n=2$

q has $n=1$

r has $n=0$, so r is a constant

Substitute $p(x)=[r(x)(x-1)+2]\cdot x+1$

$p(2)=9 \rightarrow 9=[r\cdot(2-1)+2]\cdot 2+1$

$8=[r+2]\cdot 2$

$4=r+2$

$2=r$

$$p(x) = [2(x-1)+2]\cdot x+1$$
$$= [2x-2+2]\cdot x+1$$
$$= (2x)\cdot x+1 = 2x^2+1$$

Verify: $2(0)^2+1=1$
$2(1)^2+1=3$
$2(2)^2+1=9$ ✓

2) Slightly Nastier...

$n=3$, $p(-2)=-5$, $p(0)=1$, $p(1)=4$, $p(2)=15$

$p(-2)=-5 \longrightarrow p(x)=q(x)(x+2)-5$

p has $n=3$

$p(0)=1 \longrightarrow 1=q(0)(0+2)-5$
$1=q(0)\cdot 2-5$
$6=q(0)\cdot 2$
$3=q(0)$

q has $n=2$

r has $n=1$

$q(0)=3 \longrightarrow q(x)=r(x)(x-0)+3$

Substitute. $p(x)=[r(x)\cdot x+3](x+2)-5$

$p(1)=4 \longrightarrow 4=[r(1)\cdot 1+3](1+2)-5$
$9=[r(1)+3]\cdot 3$
$3=r(1)+3$
$0=r(1)$

s has $n=0$
s is constant

$r(1)=0 \longrightarrow r(x)=s(x)(x-1)+0$
$r(x)=s\cdot(x-1)$

Substitute

$p(x) = [(s(x-1))\cdot x + 3](x+2) - 5$

$p(2) = 15 \rightarrow \quad 15 = [(s(2-1))\cdot 2 + 3](2+2) - 5$

$20 = [\ (s(1))\cdot 2 + 3](4)$

$5 = 2s + 3$

$2 = 2s$

$1 = s$

So $p(x) = [\ (x-1)\cdot x + 3](x+2) - 5$

$[x^2 - x + 3](x+2) - 5$

$x^3 + 2x^2 - x^2 - 2x + 3x + 6 - 5$

$p(x) = x^3 + x^2 + x + 1$ ✓

3) Maximum Nastiness.

$n = 4$, $(-2, 18)$ $(-1, 1)$ $(0, 4)$ $(1, 9)$ $(2, 46)$

(Give yourself a big piece of paper)

$p(-2) = 18 \rightarrow \quad p(x) = q(x)(x+2) + 18$

$p(-1) = 1 \rightarrow \quad 1 = q(-1)(-1+2) + 18$

$-17 = q(-1)$

$q(-1) = -17 \rightarrow \quad q(x) = r(x)(x+1) - 17$

Substitute.

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$(-2,18)\ (-1,1)\ (0,4)\ (1,9)\ (2,46)$

p has n=4
q has n=3
r has n=2
s has n=1

$p(x) = [r(x)(x+1) - 17](x+2) + 18$

$p(0) = 4 \rightarrow 4 = [r(0)(0+1) - 17](0+2) + 18$

$-14 = [r(0) - 17] \cdot 2$

$-7 = r(0) - 17$

$10 = r(0)$

$r(0) = 10 \rightarrow r(x) = s(x)(x-0) + 10$

Substitute.

$p(x) = [[s(x) \cdot x + 10](x+1) - 17](x+2) + 18$

$p(1) = 9 \rightarrow 9 = [[s(1) \cdot 1 + 10](1+1) - 17](1+2) + 18$

$9 = [[s(1) + 10] \cdot 2 - 17] \cdot 3 + 18$

$-9 = [[s(1) + 10] \cdot 2 - 17] \cdot 3$

$-3 = [s(1) + 10] \cdot 2 - 17$

$14 = [s(1) + 10] \cdot 2$

$7 = s(1) + 10$

$-3 = s(1)$

t is constant

$s(1) = -3 \rightarrow s(x) = t(x)(x-1) - 3$

$s(x) = t \cdot (x-1) - 3$

Substitute

$p(x) = [[(t(x-1)-3) \cdot x + 10](x+1) - 17](x+2) + 18$

$p(2) = 46$ Next page

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$$46 = [[(t(2-1)-3)\cdot 2 + 10](2+1) - 17](2+2) + 18$$
$$28 = [[(t-3)\cdot 2 + 10]\cdot 3 - 17]\cdot 4$$
$$7 = [(t-3)\cdot 2 + 10]\cdot 3 - 17$$
$$24 = [(t-3)\cdot 2 + 10]\cdot 3$$
$$8 = (t-3)\cdot 2 + 10$$
$$-2 = (t-3)\cdot 2$$
$$-1 = t-3$$
$$2 = t$$

So, $p(x) = [[(2(x-1)-3)\cdot x + 10](x+1) - 17](x+2) + 18$
$$= [[(2x-5)\cdot x + 10](x+1) - 17](x+2) + 18$$
$$= [(2x^2 - 5x + 10)(x+1) - 17](x+2) + 18$$
$$= [2x^3 - 5x^2 + 10x + 2x^2 - 5x + 10 - 17](x+2) + 18$$
$$= [2x^3 - 3x^2 + 5x - 7](x+2) + 18$$
$$= 2x^4 - 3x^3 + 5x^2 - 7x + 4x^3 - 6x^2 + 10x - 14 + 18$$
$$p(x) = 2x^4 + x^3 - x^2 + 3x + 4$$

You can verify that $p(x)$ passes through the given points.