Complex Roots #2

The Fundamental Theorem of Algebra states that every polynomial of degree n must have n roots (counting bounces, squiggles, etc.)

Sometimes the roots are complex and can be visualized on the complex plane.

<u>Ex 1</u> $x^4 - 1 = 0$ has 4 roots (degree = 4)

$x = 1$ & $x = -1$ are roots, so

$(x-1)$ & $(x+1)$ are factors. Thus,

divide by $(x-1)(x+1) = x^2 - 1$.

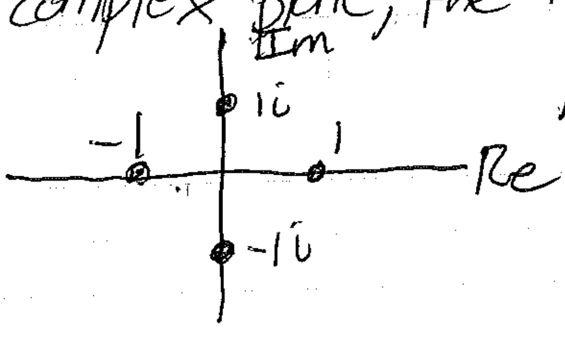
$$
\begin{array}{r|l}
 & x^2 + 1 \\
\hline
x^2 - 1 & x^4 - 1 \\
 & -(x^4 - x^2) \\
\hline
 & x^2 - 1 \\
 & x^2 - 1 \\
\hline
 & 0
\end{array}
$$

So $x^4 - 1 = (x^2+1)(x^2-1)$

Notice the difference of 2 squares!

$x^2 + 1 = 0$ if $x^2 = -1$, so $x = \pm\sqrt{-1} = \pm i$

On the complex plane, the roots are



Notice the roots are evenly spaced. The angle between is $360/4 = 90°$.

Similarly for $x^8 - 1 = 0$.

Difference of squares: $x^8 - 1 = (x^4-1)(x^4+1)$

From Ex 1 $x^4 - 1 = (x^2-1)(x^2+1)$
$(x+1)(x-1)(x+i)(x-i)$

So we need to factor $x^4+1=0$ or solve $x^4 = -1$

-1 has an angle of $180°$ on the complex plane. $180/4 = 45°$.

On the unit circle $\cos 45° = \sin 45° = \frac{\sqrt{2}}{2}$.

So the first 4th root of -1 is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

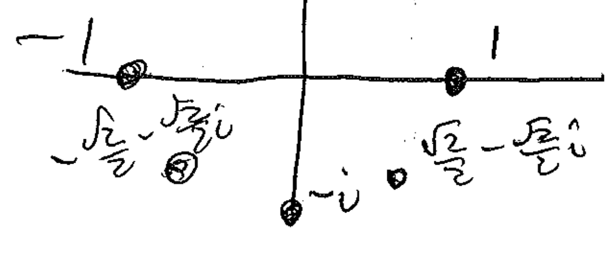
The others are evenly spaced around the circle, so $90°$ apart. That gives angles of $135°$, $225°$, & $315°$.

So my 4 roots of -1 are:

$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$
$z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

And my 8 roots of 1 are:
$z = \pm 1$, $z = \pm i$, $z = \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$

Notice they are all $360/8 = 45$ apart



More examples, less explanation

Ex) $z^2 - i = 0$

$z^2 = i$

i has angle $90°$, $90/2 = 45$

$\cos 45 = \frac{\sqrt{2}}{2}$, $\sin 45 = \frac{\sqrt{2}}{2}$, so $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

Other root is evenly spaced, so $360/2 = 180$.

$45 + 180 = 225$. $\cos 225 = \sin 225 = -\frac{\sqrt{2}}{2}$

So other $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$.

Verify by direct Computation.

$$z^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

$$\frac{2}{4} + \frac{2}{4} i + \frac{2}{4} i + \frac{2}{4} i^2$$

$$\frac{1}{2} + i - \frac{1}{2} = i$$

Ex) $z^4 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 0$

$z^4 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$-\frac{1}{2} + \frac{\sqrt{3}}{2} i$ has angle $120°$. $120/4 = 30$

$\cos 30 = \frac{\sqrt{3}}{2}$, $\sin 30 = \frac{1}{2}$, $z = \frac{\sqrt{3}}{2} + \frac{1}{2} i$

Others evenly spaced, $360/4 = 90$. So angles are $30, 120, 210, 300$

$z = \frac{\sqrt{3}}{2} + \frac{1}{2} i$, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $z = -\frac{\sqrt{3}}{2} - \frac{1}{2} i$, $z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

Verify by direct computation:

$$z^4 = \left[\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right]^2$$

$$= \left[\frac{3}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{1}{4}i^2\right]^2$$

$$= \left(\frac{3}{4} + \frac{\sqrt{3}}{2}i - \frac{1}{4}\right)^2$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \longleftarrow$$ Notice this angle is 60°. We have to get to 120° so we are halfway.

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \left(\frac{1}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\right)$$

$$= \left(\frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

In general: If $z^n = a+bi$,
find the angle of $a+bi$ & divide by n.
That is your first solution.
The others are evenly spaced, so $360/n$.
Then just do $\cos\theta = a$, $\sin\theta = b$.

Angle Addition Property: If z_1 has angle θ_1 and z_2 has angle θ_2, then $z_1 \cdot z_2$ has $\theta_1 + \theta_2$