Function composition is a fancy way of saying “plug one function into the other.”

When you have a function, “plugging in” means replacing “x” with whatever input is inside the function.

Example: $f(x) = 3x + 1$, $g(x) = 7x - 2$

$f(5) = 3(5) + 1 = 15 + 1 = 16$
$g(5) = 7(5) - 2 = 35 - 2 = 33$

Notice how all I am doing is replacing “x” with the new input of 5.

I can also have weirder inputs, and the logic is the same.

$f(1234) = 3(1234) + 1$
$g(33/7) = 7(33/7) - 2$

$f(CAT) = 3(CAT) + 1$
$g(DOG) = 7(DOG) - 2$

No matter what the input is, I can replace x with the input. Function composition is when the input is also a function. In that case, plug in the function first, then substitute the formula for the function.

$f(g(x)) = 3(g(x)) + 1 = 3(7x - 2) + 1 = 21x - 6 + 1 = 21x - 5$

Notice that $f(g(x))$ and $g(f(x))$ are different.

$g(f(x)) = 7(f(x)) - 2 = 7(3x + 1) - 2 = 21x + 7 -2 = 21x + 5$

More examples:

$h(x) = 2(x - 2)^2$ $k(x) = 3(x + 5)^2$

$h(k(x)) = 2(k(x) - 2)^2 = 2(3(x+5)^2 - 2)^2 = 2(3(x^2 + 10x + 25) - 2)^2 = 2(3x^2 + 30x + 75 - 2)^2$
$= 2(3x^2 + 30x + 73)^2$ (That seems like enough simplification)

$k(h(x)) = 3(h(x) + 5)^2 = 3(2(x - 2)^2 + 5)^2 = 3(2(x^2 - 4x + 4) + 5)^2 = 3(2x^2 - 8x + 8 +5)^2$
$= 3(2x^2 - 8x + 13)^2$

Notice again how $h(k(x))$ and $k(h(x))$ are different.

Here’s a harder example with rational functions. Notice that I multiply by the denominator (x+2), because that will cancel the fractions and let me simplify.

$$f(x)=\frac{3x+2}{x-1} \qquad g(x)=\frac{4x-1}{x+2}$$

$$f(g(x))=\frac{3(g(x))+2}{g(x)-1}=\left(\frac{3\left(\frac{4x-1}{x+2}\right)+2}{\frac{4x-1}{x+2}-1}\right)\left(\frac{x+2}{x+2}\right)$$

$$=\frac{3\left(\frac{4x-1}{\cancel{x+2}}\right)\cancel{(x+2)}+2(x+2)}{\left(\frac{4x-1}{\cancel{x+2}}\right)\cancel{(x+2)}-1(x+2)}=\frac{3(4x-1)+2(x+2)}{4x-1-1(x+2)}$$

$$=\frac{12x-3+2x+4}{4x-1-x-2}=\frac{14x+1}{3x-3}$$

$$g(f(x))=\frac{4(f(x))-1}{f(x)+2}=\left(\frac{4\left(\frac{3x+2}{x-1}\right)-1}{\frac{3x+2}{x-1}+2}\right)\left(\frac{x-1}{x-1}\right)$$

$$=\frac{4\left(\frac{3x+2}{\cancel{x-1}}\right)\cancel{(x-1)}-1(x-1)}{\left(\frac{3x+2}{\cancel{x-1}}\right)\cancel{(x-1)}+2(x-1)}=\frac{4(3x+2)-(x-1)}{3x+2+2(x-1)}$$

$$=\frac{12x+8-x+1}{3x+2+2x-2}=\frac{11x+9}{5x}$$