CCSS Advanced Algebra 4 Introduction to Complex Numbers Name ____________________

In the mid 1500s, an Italian mathematician named Gerolamo Cardano posed a simple sounding algebraic problem:

Find two numbers that add to 10 and multiply to 40.

This simple premise lead to the invention of a new set of numbers that have turned out to be interesting and useful in the development of deeper mathematical thought.

1. Let x and y be the numbers in Cardano's problem. Write two equations to represent the problem that he posed.

$x + y = 10 \longrightarrow x = (10 - y)$

$x \cdot y = 40 \qquad x \cdot y = 40 \qquad (10-y)\cdot y = 40$

2. Solve this SYSTEM OF EQUATIONS to find values of x and y. Is there a problem with the solutions you found? Explain why or why not.

$10y - y^2 = 40$

$0 = y^2 - 10y + 40$

$y = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(40)}}{2(1)}$

$y = \frac{10 \pm \sqrt{100-160}}{2} = \frac{10 \pm \sqrt{-40}}{2}$

$\sqrt{-\#}$, no real solution

3. Definitions:
   - **Imaginary Unit**: $i = \sqrt{-1}$
   - **Complex Number**: any number that can be expressed using the Imaginary Unit. For example, $5 + 2i$, $-3i$, $5 - \sqrt{15}i$.
   - **Real Part** of a Complex Number: the value of a complex number that does not include i.
   - **Imaginary Part** of a Complex Number: the value of a complex number that doe include i.

a. Use the Quadratic Formula to solve the equation $x^2 + x + 1 = 0$. Determine the Real and Imaginary parts of each solution.

$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

Real $= -\frac{1}{2}$ Imaginary $= \frac{\pm\sqrt{3}\,i}{2}$

b. Use the Quadratic Formula to solve the equation $6x^2 - 6x + 2 = 0$. Determine the Real and Imaginary parts of each solution.

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(6)(2)}}{2(6)} = \frac{6 \pm \sqrt{36-48}}{12} = \frac{6 \pm \sqrt{-12}}{12}$

Real $= \frac{6}{12} = \frac{1}{2}$, Imaginary $= \frac{\pm\sqrt{12}\,i}{12}$

c. A quadratic function has complex roots $x = 2 - i$ and $x = 2 + i$. Write the function in Standard Form. So $(x-(2-i))(x-(2+i))$ are factors.

$(x - 2 + i)(x - 2 - i) \longrightarrow$

	x	-2	$+i$
x	x^2	$-2x$	ix
-2	$-2x$	4	$-2i$
$-i$	$-ix$	$2i$	$-i^2$

$x^2 - 4x + 4 - i^2 = (x^2 - 4x + 5) \longleftarrow$

d. The polynomial function $p(x) = x^3 + 4x^2 + 11x + 8$ has a zero at $x = -1$. Find the two complex roots of $p(x)$. So $x+1$ is factor.

	x^2	$+3x$	$+8$
x	x^3	$3x^2$	$8x$
$+1$	x^2	$3x$	8

$x^2 + 3x + 8$

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(8)}}{2(1)} = \frac{-3 \pm \sqrt{9-32}}{2} = \frac{-3 \pm \sqrt{-23}}{2}$

$\frac{-3 \pm \sqrt{23}\,i}{2}$

4. Notes to Self:

- How can you use the Quadratic Formula to determine whether or not a quadratic function has complex roots?

  Look at the square root.
  If square root of negative, then cpx roots.

- How can you use the Complex Roots of a parabola to write the function in Standard Form?

  If $x=r$ is a root, then $(x-r)$ is a factor
  Use distributive property to expand to std form

- If an n^{th} degree polynomial has 2 real roots, how many complex roots must it have? If it has 3 real roots, how many complex roots must it have?

  Must have $n-2$ cpx roots
  or $n-3$ cpx roots.
  Real + Cpx = n.

5. Extra Practice:

a. Find all the roots (real and complex) of each polynomial:

i. $f(x) = x^2 + x + 11$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(11)}}{2(1)}$$
$$\frac{-1 \pm \sqrt{1-44}}{2}$$
$$\frac{-1 \pm \sqrt{43}\, i}{2}$$

ii. $g(x) = (x^2 - 4)(x^2 + x + 2)$

$(x+2)(x-2)$
$x=-2,\ x=2$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{1-8}}{2}$$
$$= \frac{-1 \pm \sqrt{7}\, i}{2}$$

iii. $h(x) = (x^2 - 2x - 8)(x^2 + 9)$

$(x-4)(x+2)$
$x=4,\ x=-2$

$x^2+9=0$
$x^2=-9$
$x=\pm 3i$

b. Solve each equation below (find real and complex solutions):

i. $-5x^2 = 3x + 10$

$-5x^2 - 3x - 10 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(-10)}}{2(-5)}$$
$$= \frac{3 \pm \sqrt{9-200}}{-10} = \frac{3 \pm \sqrt{191}\, i}{-10}$$

ii. $(x^2 + 3x + 5)(x-4)^2(2x+1) = 0$

$x=4,\ x=-1/2$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(5)}}{2(1)}$$
$$\frac{-3 \pm \sqrt{9-20}}{2}$$
$$\frac{-3 \pm \sqrt{11}\, i}{2}$$

iii. $(x^2 + 25)(x^2 - 1)(x^2 + 9) = 0$

$x = \pm 5i$, $x = \pm 1$, $x = \pm 3i$

6. Review (Rational Expressions): Simplify each expression below:

a. $\frac{3x}{x+1} + \frac{3}{x+1} = \frac{3x+3}{x+1}$

$= 3$

b. $\frac{x^2}{x+3} - \frac{9}{x+3}$

$$\frac{x^2-9}{x+3} = \frac{(x-3)\cancel{(x+3)}}{\cancel{x+3}}$$

c. $\frac{1}{x-1} + \frac{1}{x+1} = \left(\frac{x+1}{x+1}\right)\frac{1}{x-1} + \frac{1}{x+1}\left(\frac{x-1}{x-1}\right)$

$$\frac{x+1+x-1}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

d. $\frac{\cancel{x}}{\cancel{x-3}} \cdot \frac{\cancel{(x-3)}(x+3)}{\cancel{x^2}} = \frac{x+3}{x}$

e. $\frac{(x^2-25)(x^2+6x+5)(x-1)}{(x^2+10x+25)(x^2-1)}$

$$= \frac{\cancel{(x+5)}(x-5)\cancel{(x+5)}\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+5)}\cancel{(x+5)}\cancel{(x+1)}\cancel{(x-1)}} = x-5$$